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Tsallis' q-triplet and the ozone layer

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ABSTRACT

Tsallis' q-triplet [C. Tsallis, Dynamical scenario for nonextensive statistical mechanics, Physica A 340 (2004) 1–10] is the best empirical quantifier of nonextensivity. Here we study it with reference to an experimental time-series related to the daily depth-values of the stratospheric ozone layer. Pertinent data are expressed in Dobson units and range from 1978 to 2005. After the evaluation of the three associated Tsallis' indices one concludes that nonextensivity is clearly a characteristic of the system under scrutiny.

1. Introduction

1.1. Preliminaries

Stratospheric ozone is concentrated mainly within a ~15 km layer at a height of about 15 km, with a very low density of a few O_3-molecules per million air molecules. It is well known that these molecules are essential to terrestrial life because of their blocking effect with regard to UV radiation. The detailed mechanism of the pertinent chemical interactions is described in Ref. [1]. The short-range variability of the ozone layer's depth over a given surface spot is influenced in a highly nonlinear fashion by (i) the solar radiation intensity and (ii) weather changes. As a consequence, thermal equilibrium is never achieved. A stationary regime is reached that is modulated by several kinds of oscillations, namely, (1) a yearly one due to the orientation of the incoming radiation, (2) another one of a period of almost 2 years due to stratospheric air-currents, and (3) a secular one [1]. In this work we focus attention on two time-series: (a) $\{Z_n\}$ of depth-values for the ozone layer and (b) its daily variability $\{\Delta Z_n\}$. We employ to this effect several tools pertaining to the field of nonextensive statistical mechanics [2].

1.2. Special features of nonextensive statistical mechanics (NST)

NST was advanced by Tsallis en 1988 [3,2], as a generalization of the conventional, additive Boltzmann–Gibbs (BG), statistical mechanics (SM) so that some of its tools could now be used for systems unaccessible before. A particularly important instance is that of systems that find themselves in nonequilibrium but still stationary states, found in a great

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variety of complex systems. The main SM-ingredient is the microscopic expression for the entropy in terms of microstates' (labelled by, say, i) probabilities p_i. Instead of the BG logarithmic relation $S_{BG} = -k\sum_i p_i \ln p_i$ Tsallis proposes $S_q = -k(1-\sum_i p_i^q)/(1-q)$; $(q \in \mathcal{R})$, with k standing for Boltzmann's constant. For $q \to 1$, $S_q \to S_{BG}$. q is usually referred to as the nonextensivity index.

The BG, conventional SM exhibits three main features.

- It leads to a probability distribution function (PDF) describing thermal equilibrium of exponential form in the energy. Thus, the probability of detecting the system in a state of energy de u is es proportional to $\exp(-\beta u)$.
- Systems well-described à la BG exhibit exponential sensibility to initial conditions. Small initial differences between neighboring states grow in exponential fashion (chaotic dynamics characterized by one or more positive Lyapunov exponents).
- Macroscopic variables exponentially decay to their equilibrium values with a relaxation time τ.

Analogously, Tsallis (or q-) statistics exhibits the following three counterpart-features [4], namely:

(i) Tsallis (or q-) PDF's, that describe meta-stable or stationary states, are proportional to functions called q-exponentials, defined according to

$$\exp_q(-\beta u) = [1-(1-q)\beta u]^{1/(1-q)}, \tag{1}$$

with β and q constants. In the limit $q \to 1$ q-exponentials become ordinary ones ($\exp_1(x) \equiv \exp(x)$). If $q \to 1$ and $u = y^2$, $\exp_q(-\beta u)$ becomes a q-Gaussian. Stationary states are characterized by a parameter $q \equiv q_{\mathrm{stat}}$. The inverse of the q-exponential is the so-called q-logarithm

$$\ln_q(x) = \frac{x^{1-q}-1}{1-q}, \qquad \ln_1(x) = \ln(x), \qquad \ln_q[\exp_q(x)] = \exp_q[\ln_q(x)] = 1. \tag{2}$$

(ii) Stationary states exhibit q-exponential sensibility to initial conditions ("weak" chaos). Small initial differences between neighboring states grow in q-exponential fashion, characterized by a vanishing Lyapunov exponent and a parameter q_{sens}.

(iii) Macroscopic variables q-exponentially decay to their stationary values with $q = q_{\mathrm{rel}}$.

Accordingly, a stationary or meta-stable state is characterized by a triplet of q-values (the Tsallis' "q-triplet"), i.e., $(q_{\mathrm{stat}}, q_{\mathrm{sens}}, q_{\mathrm{rel}}) \neq (1, 1, 1)$, where $q_{\mathrm{stat}} > 1$, $q_{\mathrm{sen}} < 1$, and $q_{\mathrm{rel}} > 1$ [4]. Since the publication date of [4] till now, as far as we know, just one *empirical* instance of triplet detection has been reported, with reference to the distant helio-magnetic field intensity [5,6]. Values for the particular parameter q_{stat} have been reported for the daily variation of the ENSO-index (El Niño Southern Oscillation) [7], and for fluctuations of the cosmic back-ground radiation [8].

2. The data

The terrestrial ozone layer is monitored daily via the NASA satellite-project "Total Ozone Metering Spectrometer" [1] since 1978, so that long time-series are available [9]. Layer's depth-values are expressed in Dobson units (DU). 100DU are equivalent to a 1 mm-height column for a gas at atmospheric pressure. We will employ satellite data corresponding to Buenos Aires city. These are daily values obtained since Nov. '78 till May '93 and from July '96 till Dec. '05. Due to the manifold factors influencing the ozone layer's depth, we decompose our series according to (see the Introduction)

Original data = mean value + long range tendency + annual oscillation + quasi-biannual one + Z_n.

A mean-square adjustment of the original series is made via a linear function plus two periodic functions (1 and 2 years' periods) [1] resulting in a function H. Afterwards, we perform a subtraction: from the original series we deduct the H-values, yielding a new series Z_n from which deterministic oscillation have been eliminated. Z_n is the system's attractor, expressing the nonlinear dynamics induced by climatic changes and daily variability of the solar radiation (see Fig. 1).

3. Results and discussion

3.1. Stationary q

q-Gaussians generalize normal Gaussian distributions [10], i.e.,

$$G_q(\beta; z) = \frac{\sqrt{\beta}}{C_q} \mathrm{e}_q^{-\beta z^2}, \quad 1 < q < 3, \; C_q = \frac{\sqrt{\pi}\,\Gamma(\frac{3-q}{2(q-1)})}{\sqrt{q-1}\,\Gamma(\frac{1}{q-1})}. \tag{3}$$

The suitable q-value for the stationary is obtained from the PDF associated to daily variations of the ozone layer's depth $\Delta Z_n = Z_{n+1} - Z_n$. The ΔZ-range is subdivided into little "cells" of width δz centered at z_i so that one can assess with what frequency ΔZ-values fall within each cell. We chose a cell-size $\delta z = 5UD$. The resultant histogram, properly normalized, yields our stationary-PDF $\{p(z_i)\}_{i=1}^N$. Of course, p_i is the probability for a ΔZ-value to fall within the ith cell, centered at z_i,

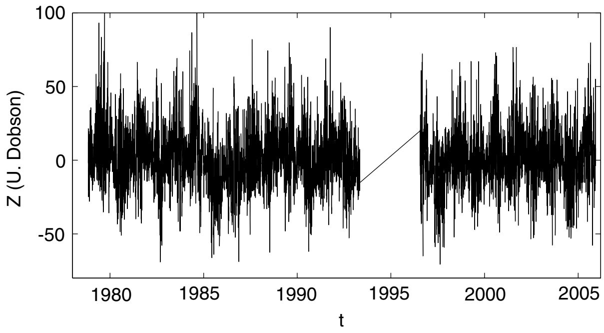

Fig. 1. Time-series Z_n. Daily values of the ozone layer over Buenos Aires city.

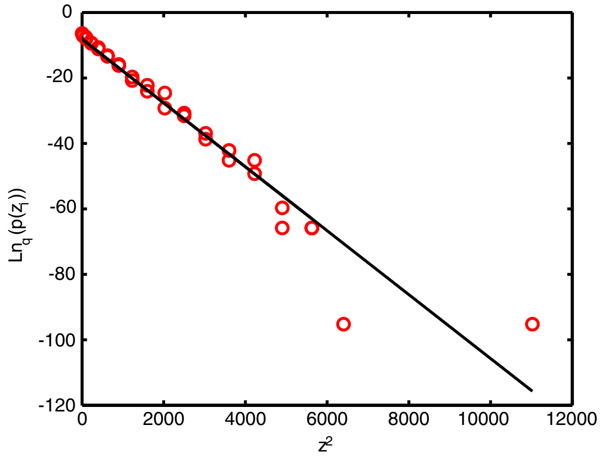

Fig. 2. Linear correlation between $\ln_q[p(z_i)]$ and z_i^2, where $q_{\text{stat}} = 1.32 \pm 0.06$. The linear CC. is 0.9383.

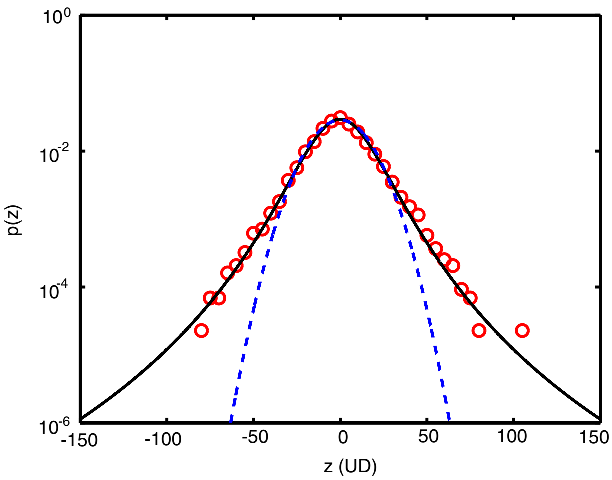

Fig. 3. In red circles, $p(z_i)$ vs. z_i; in solid black line, the q-Gaussian function that fits $p(z_i)$, Eq. (3), with $\beta = 0.00356$; in dashed blue line, the best adjustment with a *Gaussian*. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

with N the cell-number. The graph $\ln_q[p(z_i)]$ vs. z_i^2 becomes of interest here. For a proper assessment we varied q within $[1, 1.5]$ with $\delta_q = 0.01$, making in each instance a linear adjustment and evaluating the associated correlation coefficient (CC). The best CC obtains for $q_{\text{stationary}} = 1.32 \pm 0.06$, with CC $r^2 = 0.9383$ (see Fig. 2).

With such a q-value one computes then Eq. (3) at z_i for different β-values, selecting the one minimizing $\sum_i [G_{1.324}(\beta; z_i) - p(z_i)]^2$. The q-Gaussian with $\beta = 0.00356 UD^{-2}$ is depicted in Fig. 3 (continuous black line). The best adjustment that can be

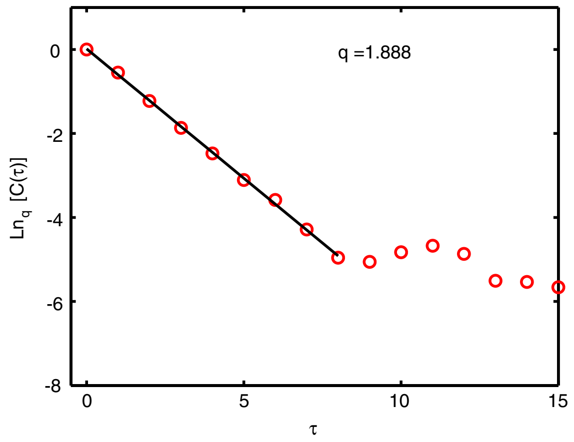

Fig. 4. $\ln_q$ of the self-correlation coefficient $C(\tau)$ vs. time delay τ (in days). We obtain the best fit with $q_{rel} = 1.888$. The linear CC is 0.99919.

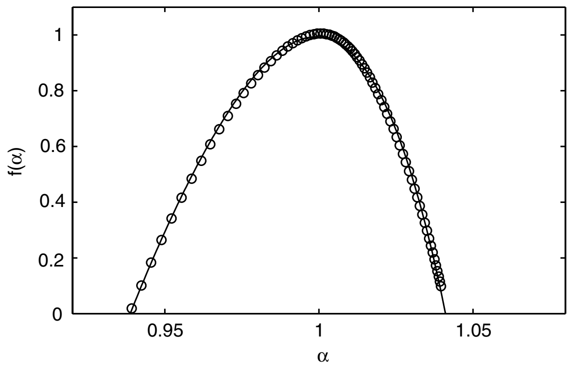

Fig. 5. Multifractal spectrum of ozone time series. $\alpha_{min} = 0.938 \pm 0.001$ and $\alpha_{max} = 1.046 \pm 0.001$.

obtained with a *Gaussian* is also plotted (blue dashed line). It is clear that, for $|z| \geq 20UD$, $p(z_i)$-values become noticeably nonGaussian. Our function $G_{1.32}(0.00356; z)$ fits quite instead the $p(z_i)$ for all z.

3.2. *Relaxation q*

The corresponding q_{rel}-value is determined via the temporal self-correlation coefficient

$$C(\tau) = \frac{\sum_n Z_{n+\tau}.Z_n}{\sum_n Z_n^2}. \tag{4}$$

For a classical BG-process such correlation should decay in exponential fashion. However we do not find such a behavior. Instead, the self-correlation of our series Z_n clearly decays from unity in a q-exponential manner, with a $q = q_{rel} = 1.89 \pm 0.02$ (see Fig. 4).

3.3. *Sensibility to initial conditions $q = q_{sens}$*

q_{sens} can be derived from the multifractal spectrum $f(\alpha)$ of the attractor associated to our nonlinear dynamical system, reflected by Z_n. $f(\alpha)$ is the fractal dimension of the subset of points of the attractor that possesses the local scaling index α [12]. The spectrum's extremes α_{min} and α_{max}, for which $f(\alpha) = 0$, are related to q_{sens} [4,11] according to

$$\frac{1}{(1 - q_{sens})} = \frac{1}{\alpha_{min}} - \frac{1}{\alpha_{max}}. \tag{5}$$

We have evaluated the multifractal spectrum for the time series as explained in Refs. [13,12] and can be appreciated in Fig. 5. We assign a value to q_{sens} via spectrum's extrapolation with a fourth degree polynomial so as to determine $\alpha_{min} = 0.938 \pm 0.001$ and $\alpha_{max} = 1.046 \pm 0.001$, thus we have $q_{sens} = -8.1 \pm 0.2$. Unfortunately, Tsallis relations among our three "q"-values [2] similar to those pertaining to the solar wind ones cannot be obtained in the present instance. Notice however that our three "q"-values do verify

$$q_{\text{sens}} < 1 < q_{\text{stat}} < q_{\text{rel}},$$

as pointed out in page 308 of Ref. [2].

4. Conclusions

We worked with NASA's time-series for the ozone layer's depth at Buenos Aires city. With such data we were able to estimate the Tsallis-nonextensivity triplet.

The three associated indices adopt the values

$$\{q_{\text{stat}}; q_{\text{sens}}; q_{\text{rel}}\} = \{1.32 \pm 0.06; -8.1 \pm 0.2; 1.89 \pm 0.02\}, \quad (6)$$

with [2] clearly indicating that we are in the presence of a system in an off-equilibrium stationary state whose physics is properly described by the q-statistical mechanics.

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