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ABSTRACT

In the realm of Boltzmann–Gibbs statistical mechanics, there are three well known isomorphic connections with random geometry, namely, (i) the Kasteleyn–Fortuin theorem, which connects the $\lambda \rightarrow 1$ limit of the λ -state Potts ferromagnet with bond percolation, (ii) the isomorphism, which connects the $\lambda \rightarrow 0$ limit of the λ -state Potts ferromagnet with random resistor networks, and (iii) the de Gennes isomorphism, which connects the $n \rightarrow 0$ limit of the n -vector ferromagnet with self-avoiding random walk in linear polymers. We provide here strong numerical evidence that a similar isomorphism appears to emerge connecting the energy q -exponential distribution $\propto e_q^{-\beta q \mathcal{E}}$ (with $q = 4/3$ and $\beta_q \omega_0 = 10/3$) optimizing, under simple constraints, the nonadditive entropy S_q with a specific geographic growth random model based on preferential attachment through exponentially distributed weighted links, ω_0 being the characteristic weight.

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Several examples exist of isomorphism between specific models in the realm of Boltzmann–Gibbs statistical mechanics with random geometry models. Such examples include the Kasteleyn–Fortuin theorem related to bond percolation, the zero-state limit of the Potts ferromagnet related to random resistor networks, and the de Gennes isomorphism of the zero-component limit of the n -vector model with self-avoiding random walk. We present here strong numerical evidence that the same happens in the realm of nonextensive statistical mechanics. Indeed, the energy distribution associated with a geographical d -dimensional preferential-attachment-based (asymptotically) scale-free growth model is given by a simple q -exponential with $q = 4/3$.

I. INTRODUCTION

Within Boltzmann–Gibbs (BG) statistical mechanics, the energy distribution of a Hamiltonian system in thermal equilibrium

at temperature T is given by the following matrix density ρ :^{1,2}

$$\rho(\mathcal{H}) = \frac{e^{-\beta \mathcal{H}}}{Z(\beta)}, \quad (1)$$

where $\beta \equiv 1/kT$ and the partition function is defined as $Z(\beta) \equiv \text{Tr} e^{-\beta \mathcal{H}}$, \mathcal{H} being the Hamiltonian. In their diagonalized version, these quantities become

$$p(E_i) = \frac{e^{-\beta E_i}}{Z(\beta)}, \quad (2)$$

where $Z(\beta) \equiv \sum_i e^{-\beta E_i}$, E_i being the i th eigenvalue of the Hamiltonian \mathcal{H} .

Occasionally, there are models whose BG thermostistical approach is isomorphic to random geometrical models. Let us mention three well known such examples: (i) The Kasteleyn–Fortuin theorem,³ which connects bond percolation in an arbitrary graph or lattice with the $\lambda \rightarrow 1$ limit of the λ -state Potts ferromagnetic

model in the same graph (see Ref. 4 and references therein). (ii) The random resistor (or impedance) network, which connects the Ohmic behavior of an arbitrary resistor graph with the $\lambda \rightarrow 0$ limit of the just mentioned λ -state Potts ferromagnetic model in the same graph (see Refs. 4 and 5 and references therein). (iii) The de Gennes isomorphism^{6,7} which connects self-avoiding random walk on an arbitrary graph (equivalently the growth configurations of a linear polymer) with the $n \rightarrow 0$ limit of the n -vector ferromagnet on the same graph. We briefly review these three isomorphisms in Sec. II.

In the present paper (Sec. III), we describe and numerically study a random geometrical model, namely, the growth of an asymptotically scale-free geographic weighted-link preferential-attachment network. Its numerical study provides a strong indication of being isomorphic to a simple thermostistical model within nonextensive statistical mechanics.^{8–10}

II. ISOMORPHIC MODELS WITHIN THE BG STATISTICAL MECHANICS

We briefly review here three well known examples of isomorphism between models within BG thermostistics and nontrivial random geometrical models. We refer to the $\lambda \rightarrow 1$ and $\lambda \rightarrow 0$ limits of the λ -state Potts ferromagnet and the $n \rightarrow 0$ limit of the n -vector ferromagnet.

A. The Potts ferromagnet

The Hamiltonian of the λ -state Potts ferromagnetic model¹¹ is defined as follows:

$$\mathcal{H} = -\lambda \sum_{ij} J_{ij} \delta_{\sigma_i, \sigma_j} \quad (\sigma_i = 1, 2, \dots, \lambda, \forall i), \quad (3)$$

where $J_{ij} > 0, \forall(i, j)$, and the sum runs over all pairs of “spins” located at the sites of an arbitrary lattice (finite or infinite, regular or not, translationally invariant, i.e., Bravais lattice, scale invariant, i.e., fractal lattice, etc.) and $\delta_{\sigma_i, \sigma_j}$ is Kronecker’s delta. The particular case $\lambda = 2$ is (through a trivial energy shift in the Hamiltonian) identical to the standard spin 1/2 Ising model. The elementary Potts interaction (single bond) yields a two-level spectrum: one level with energy $-\lambda J_{ij}$ and degeneracy λ and the other one with energy 0 and degeneracy $\lambda(\lambda - 1)$. To every single bond between sites i and j , we may associate its *thermal transmissivity* (see Ref. 4 and references therein),

$$t_{ij} \equiv \frac{1 - e^{-\lambda J_{ij}/kT}}{1 + (\lambda - 1)e^{-\lambda J_{ij}/kT}} \in [0, 1]. \quad (4)$$

Let us first consider a *series* array of two bonds or links (with Potts coupling constants J_1 and J_2) and three vertices or sites. The transmissivity after tracing over the λ states of the internal site is given by

$$t_s = t_1 t_2 \text{ (series)}. \quad (5)$$

If we have instead a *parallel* array of two bonds (again with Potts coupling constants J_1 and J_2), the resulting coupling constant is

given by $J_1 + J_2$, which straightforwardly leads to

$$t_p = \frac{t_1 + t_2 + (\lambda - 2)t_1 t_2}{1 + (\lambda - 1)t_1 t_2} \text{ (parallel)} \quad (6)$$

(or, equivalently, $\frac{1-t_p}{1+(\lambda-1)t_p} = \frac{1-t_1}{1+(\lambda-1)t_1} \frac{1-t_2}{1+(\lambda-1)t_2}$). If we have two-open-site arrays that are not reducible to sequences of series and parallel operations (e.g., the Wheatstone bridge), we may use the break-collapse method (see Ref. 4 and references therein) to calculate the equivalent transmissivity.

1. The $\lambda \rightarrow 1$ limit

If we consider the analytic $\lambda \rightarrow 1$ limit, Eq. (5) remains as it stands, whereas Eq. (6) becomes

$$t_p = t_1 + t_2 - t_1 t_2 \text{ (parallel)} \quad (7)$$

[or, equivalently, $(1 - t_p) = (1 - t_1)(1 - t_2)$]. We notice that Eqs. (5) and (7) are precisely the composition laws of independent probabilities. This is the basis of the Kasteleyn–Fortuin theorem,³ which rigorously establishes the isomorphism of the $\lambda \rightarrow 1$ Potts ferromagnet in an arbitrary lattice with bond percolation in the same lattice.

2. The $\lambda \rightarrow 0$ limit

If we define now

$$t_i \equiv 1 - \frac{g_0}{g_i}, \quad (8)$$

where g_0 is some reference electrical conductance (i.e., the inverse of a reference electrical resistance), we straightforwardly verify that, in the $g_0/g_i \rightarrow 0$ limit, Eqs. (5) and (6) become, respectively, (see, for instance, Ref. 5)

$$g_s = \frac{g_1 g_2}{g_1 + g_2} \text{ (series)} \quad (9)$$

(or, equivalently $\frac{1}{g_s} = \frac{1}{g_1} + \frac{1}{g_2}$) and

$$g_p = g_1 + g_2 \text{ (parallel)}. \quad (10)$$

These composition laws are precisely those of Ohmic conductances. If we have two-open-site arrays that are not reducible to sequences of series and parallel operations, we may use once again the break-collapse method⁵ to calculate the equivalent conductance. This constitutes the basis of the isomorphism of the $\lambda \rightarrow 0$ Potts ferromagnet in an arbitrary lattice with random resistors in the same lattice.

B. The n -vector ferromagnet

The Hamiltonian of the n -vector or the $O(n)$ ferromagnetic model can be defined as follows:¹²

$$\begin{aligned} \mathcal{H} &= - \sum_{ij} J_{ij} \vec{s}_i \cdot \vec{s}_j \\ &= - \sum_{ij} J_{ij} \sum_{k=1}^n s_i^{(k)} s_j^{(k)} \left(\sum_{k=1}^n [s_i^{(k)}]^2 = 1, \forall i \right), \end{aligned} \quad (11)$$

where $J_{ij} > 0, \forall(i, j)$, and the first sum runs over all pairs of spin located at the sites of an arbitrary lattice (finite or infinite, regular

or not, translationally invariant, i.e., Bravais lattice, scale invariant, i.e., fractal lattice, etc.). The particular case $n = 1$ corresponds to the Ising model; $n = 2$ corresponds to the XY model; $n = 3$ corresponds to the Heisenberg model; $n \rightarrow \infty$ corresponds to the spherical model. In 1972, de Gennes proved⁶ that the analytical extension $n \rightarrow 0$ is isomorphic to the growth of a self-avoiding linear polymer in the same lattice. This isomorphism certainly constitutes one of the landmarks of polymer physics.

III. POSSIBLE ISOMORPHIC MODEL WITHIN NONEXTENSIVE STATISTICAL MECHANICS

A. The random geometric weighted-link preferential-attachment growth model

The growing d -dimensional network we focus on here has been introduced and studied in Ref. 13, which we follow now. We start with one site at the origin. We then stochastically locate a second site (and then a third, a fourth, and so on up to N) through a probability $p(r) \propto 1/r^{d+\alpha_G}$ ($\alpha_G > 0$), where $r \geq 1$ is the Euclidean distance from the newly arrived site to the center of mass of the pre-existing cluster; α_G is the *growth* parameter and $d = 1, 2, 3$ is the dimensionality of the system (large α_G yields geographically concentrated networks).

The site $i = 1$ is then linked to the site $j = 2$. We sample a random number w_{ij} from a distribution $P(w)$ that will give us the corresponding link weight. Each site will have a *total energy* ε_i that will depend on how many links it has, noted k_i , and the widths $\{w_{ij}\}$ of those links. At each time step, the site i only has access to its *local energy* ε_i defined as

$$\varepsilon_i \equiv \sum_{j=1}^{k_i} \frac{w_{ij}}{2} \quad (w_{ij} \geq 0). \tag{12}$$

The value of ε_i will directly affect the probability of the site i to acquire new links. Indeed, from this step on, the sites $i = 3, 4, \dots$ will be linked to the previous ones with probability

$$\Pi_{ij} \propto \frac{\varepsilon_i}{d_{ij}^{\alpha_A}} \quad (\alpha_A \geq 0), \tag{13}$$

where d_{ij} is the Euclidean distance between i and j , where j runs over all sites linked to the site i . The *attachment* parameter α_A controls the importance of the distance in the preferential attachment rule (13). When $\alpha_A \gg 1$, the sites tend to connect to close neighbors, whereas $\alpha_A \simeq 0$ tends to generate distant connections all over the network. Notice that, while the network size increases up to N nodes, the variables k_i and ε_i (number of links and *total energy* of the i -th node; $i = 1, 2, 3, \dots, N$) also increase in time.

If we consider the particular case $P(w) = \delta(w - 1)$, where $\delta(z)$ denotes the Dirac delta distribution, Eq. (13) becomes $\Pi_{ij} \propto k_i/d_{ij}^{\alpha_A}$ ($\alpha_A \geq 0$), thus recovering the usual preferential attachment rule (see, for instance, Refs. 14 and 15 and references therein). Note that if we additionally consider the particular case $\alpha_A = 0$, we recover the standard Barabási–Albert model with $\Pi_i \propto k_i$.^{16,17}

We are considering here the case where w is given by the following stretched-exponential distribution:

$$P(w) = \frac{\eta}{w_0 \Gamma(\frac{1}{\eta})} e^{-(w/w_0)^\eta} \quad (w \geq 0; w_0 > 0; \eta > 0), \tag{14}$$

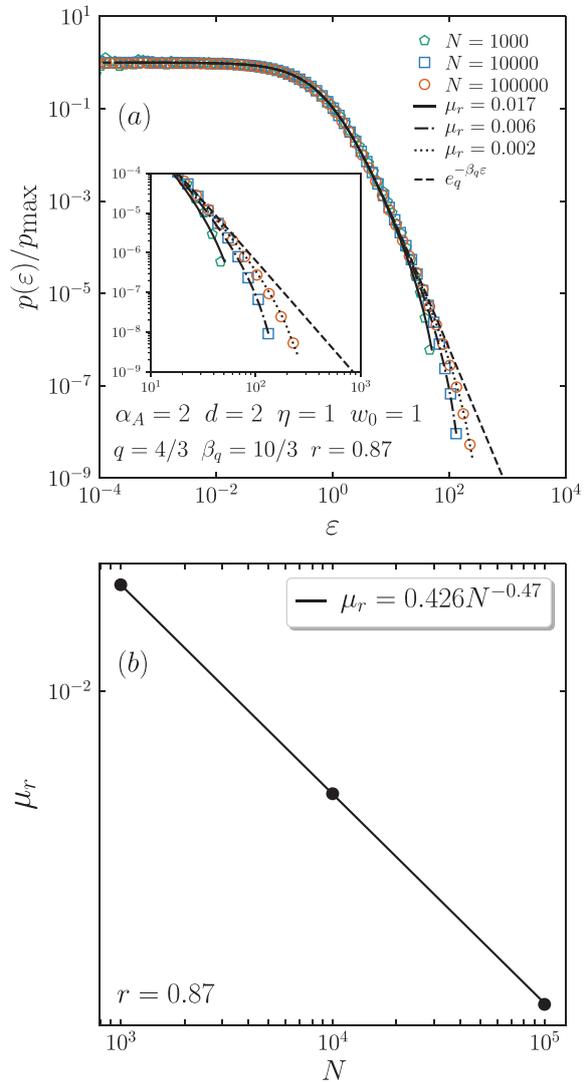


FIG. 1. (a) The ε -dependence of the normalized probability $\xi(\varepsilon) \equiv p(\varepsilon)/p(0)$ corresponding to $(\eta, w_0, \alpha_A, d) = (1, 1, 2, 2)$ and three typical values of N . The curves joining the points indicated for each value of N are produced using $\xi(\varepsilon)$ given by Eq. (16) with $r = 0.87$ and the corresponding values of μ_r . The dashed curve corresponds to the conjectured $N \rightarrow \infty$ limit $\xi(\varepsilon) = e_q^{-\beta_q \varepsilon}$ with $(q, \beta_q) = (4/3, 10/3)$. (b) The N -dependence of μ_r , where the fitting parameter r has been chosen so that a straight line emerges in a log-log representation, precisely corresponding here to $\mu_r = 0.426 N^{-0.47}$.

which satisfies $\int_0^\infty dw P(w) = 1$. As particular cases of Eq. (14), we have $\eta = 1$, which corresponds to an exponential distribution; $\eta = 2$, which corresponds to a half-Gaussian distribution; and $\eta \rightarrow \infty$, which corresponds to a uniform distribution within

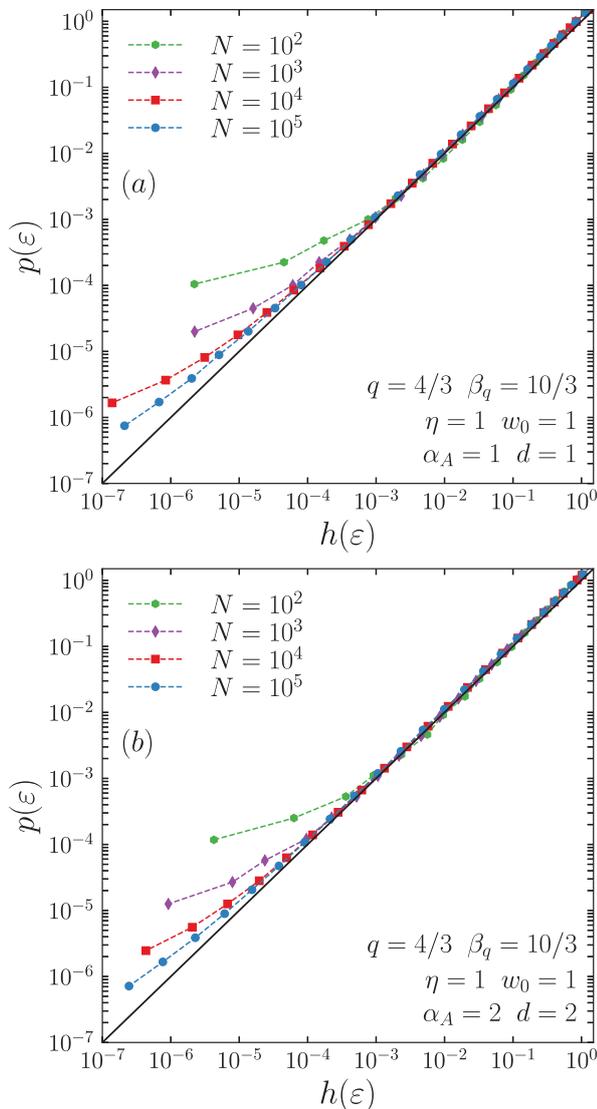


FIG. 2. The probability $p(\varepsilon)$ as a function of the corresponding histogram $h(\varepsilon)$ for $(\eta, w_0) = (1, 1)$ and typical values of N with $\alpha_A = d = 1$ (a) and $\alpha_A = d = 2$ (b). Comparison of the histogram $h(\varepsilon)$ of site energies with the nearly q -exponential distribution $p(\varepsilon)$. The dashed lines are a guide to the eye. The bisector straight line corresponds to the conjectured $N \rightarrow \infty$ limit, namely, $p(\varepsilon) = p(0) e_q^{-\beta_q \varepsilon}$ with $(q, \beta_q) = (4/3, 10/3)$.

$w \in [0, w_0]$. Our aim here is to specifically study the $\eta = 1$ case; i.e., $p(w) = w_0^{-1} e^{-w/w_0}$ ($w \geq 0; w_0 > 0$). In Fig. 1, we present numerical results for $\alpha_A = d = 2$ and increasing values of N . These results remain, in fact, the same for any (α_A, d) values such that $0 \leq \alpha_A/d \leq 1$. This is illustrated in Fig. 2 for $\alpha_A/d = 1$ ($d = 1, 2$).

B. Finite-size effects

As we verify in both Figs. 1 and 2, there are sensible effects on $p(\varepsilon)$ coming from the finiteness of N , which seemingly disappear in the $N \rightarrow \infty$ limit. Following along the lines of Ref. 18, we check that these finite-size effects are satisfactorily described by the following equation:

$$\frac{d\xi}{d\varepsilon} = -\mu_r \xi^r - (\beta_q - \mu_r) \xi^q \quad (r \leq q; \varepsilon \geq 0), \tag{15}$$

where $\xi(\varepsilon) \equiv p(\varepsilon)/p^{\max} \in [0, 1]$. Consequently,

$$\begin{aligned} \varepsilon &= \int_{\xi}^1 \frac{dx}{\mu_r x^r + (\beta_q - \mu_r) x^q} \\ &= \frac{1}{\mu_r} \int_{\xi}^1 dx \left\{ \frac{1}{x^r} - \frac{(\beta_q/\mu_r - 1)x^{q-2r}}{1 + (\beta_q/\mu_r - 1)x^{q-r}} \right\} \\ &= \frac{1}{\mu_r} \left\{ \frac{\xi^{1-r} - 1}{r-1} - \frac{(\beta_q/\mu_r) - 1}{1+q-2r} \right. \\ &\quad \times [H(1; q-2r, q-r, (\beta_q/\mu_r) - 1) \\ &\quad \left. - H(\xi; q-2r, q-r, (\beta_q/\mu_r) - 1)] \right\}, \tag{16} \end{aligned}$$

with

$$H(\xi; a, b, c) \equiv \xi^{1+a} F\left(\frac{1+a}{b}, 1; \frac{1+a+b}{c}; -\xi^b c\right), \tag{17}$$

where F is the hypergeometric function. We can verify, for instance, that, for $1 < r < q$ and $0 < \mu_r \ll \beta_q$, three regions emerge. These three regions are characterized as follows: for small ε , ξ is nearly constant; for intermediate ε , ξ decreases as $1/\varepsilon^{1/(q-1)}$; finally, for large ε , ξ further decreases, now as $1/\varepsilon^{1/(r-1)}$. If $r \leq 1$, this function vanishes even faster for increasing ε (see details in Ref. 18).

IV. CONCLUSION

The present numerical results provide a strong indication that, in the $N \rightarrow \infty$ limit, μ_r vanishes. Consequently, the differential equation $\frac{d\xi}{d\varepsilon} = -\beta_q \xi^q$ ($\varepsilon \geq 0$) is expected to be satisfied; hence,

$$p(\varepsilon) = p(0) e_q^{-\beta_q \varepsilon}. \tag{18}$$

Let us emphasize at this point that this distribution precisely is the one that optimizes, under simple constraints, the nonadditive entropy $S_q = k \frac{1 - \sum_i p_i^q}{q-1}$ ($S_1 = S_{BG} = -k \sum_i p_i \ln p_i; \sum_i p_i = 1$).⁸⁻¹⁰

From the discussion presented in Fig. 2, as well as a variety of numerical checks that we have concomitantly performed, we are allowed to conjecture that $(q, \beta_q) = (4/3, 10/(3w_0))$ for $\alpha_A/d \in [0, 1]$. Therefore, the $N \rightarrow \infty$ limit of the random geometry growth model that we have focused on here appears to be isomorphic to a simple model within nonextensive statistical mechanics, more precisely the model whose total energy is just the sum of all the site energies $\{\varepsilon_i\}$. This connection obviously is fully analogous to those three described in Sec. II within Boltzmann-Gibbs statistical mechanics. Its rigorous proof remains to be done.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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